From Kaluza-Klein theory to Campbell-Magaard theorem and beyond

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Abstract

We give a brief review of recent developments in five-dimensional theories of spacetime and highlight their geometrical structure mainly in connection with the Campbell-Magaard theorem.

> *"Ainsi, dans la cosmologie la géometrie s'organise*
>
> *dans un cadre idéalisé où elle apparaît consubstantielle*
>
> *à un ordre préfabriqué du monde."*
>
> *(M.Novello, Cosmos et Contexte,1987)*

I. KALUZA-KLEIN THEORY

The idea that our spacetime might have five dimensions and that some observable physical effects could be attributed to the existence of a fifth dimension has perhaps no better illustration than that provided by the Kaluza-Klein theory [1,2]. Indeed, in this theory a scheme is devised in which the extra dimensionality of space is combined with curvature in

such a way that electromagnetic phenomena may be looked upon as a pure manifestation of geometry. Although no new physical effect was to be predicted by the theory, the very possibility of unification (even at a mathematical level) between two different subjects such as gravity and electromagnetism has had an appeal which endures until today. Of course along the history of physics there have been many attempts to formulate a unified field theory which would not resort to extra dimensions. In fact, the fifth dimension assumption was basically the cause of Einstein's reluctance to accept the plausibility of Kaluza-Klein theory [3]. Alternative ways to geometrize the electromagnetic field in the usual four-dimensional spacetime may be found, for instance, in the work of Weyl, who proposed a generalization of Riemannian geometry [4]. More recently, Weyl's approach has been revived by Novello through the WIST (Weyl Integrable Space-Time) program, where interesting applications to Cosmology have been discussed [5]. The development of particle physics, on the other hand, led to a ressurgence of interest in higher-dimensional field theories, mainly as a possibility of unifying the long-range and short-range interactions. Indeed, inspired by the old five-dimensional Kaluza-Klein theory there appeared, around the sixties and seventies of the last century, higher-dimensional theories such as eleven-dimensional supergravity and ten-dimensional superstrings, all aiming at a unifying scheme [6,7].

II. INDUCED MATTER THEORY

The original version of Kaluza-Klein theory assumes as a postulate that the fifth dimension is compact (condition of cilindricity). Recently, however, a non-compactified approach to Kaluza-Klein gravity, known as Induced-Matter theory (IMT) has been proposed by Wesson and colaborators [8]. The basic principle of the IMT approach is that all classical physical quantities, such as matter density and pressure, should be given a geometrical interpretation [9]. Moreover, it is asserted that only one extra dimension should be sufficient to explain all the phenomenological properties of matter. More specifically, IMT proposes that the classical energy-momentum tensor, which enters the right-hand side of the Einstein

equations could be, in principle, generated by a pure geometrical means. The theory also assumes that the fundamental five-dimensional space M^5, in which our usual spacetime is embedded, should be a solution of the five-dimensional vacuum Einstein equations

$$R_{\alpha\beta} = 0.$$

We shall not go into the details of the mathematical mechanism of Induced-Matter theory (the interested reader is referred to [8,9] and the references therein). For our purposes here, suffice it to say that IMT has a mathematical structure which can be better understood if it is regarded as a spacetime embedding theory [10]. In this context our spacetime would correspond to a four-dimensional hypersurface locally and isometrically embedded in a five-dimensional Ricci-flat manifold.

III. CAMPBELL-MAGAARD THEOREM

Embedding theories are naturally subject to embedding theorems of differential geometry. The claim that any energy-momentum can be generated by an embedding mechanism may be translated in geometrical language as saying that any semi-Riemannian four-dimensional manifold is embeddable into a five-dimensional Ricci-flat manifold. At the time IMT was proposed it was not at all apparent whether such proposition was true. Fortunately (at least for the supporters of the theory), this is essentially the content of a little known but powerful theorem due to Campbell [11] and Magaard [12], which asserts that any semi-Riemannian n-dimensional analytic manifold can be locally and isometrically embedded in a semi-Riemannian (n+1)-dimensional analytic manifold, where the Ricci tensor of the latter vanishes [13]. Campbell-Magaard's result has then acquired fundamental relevance for granting the mathematical consistency of five-dimensional non-compactified Kaluza-Klein gravity.

Local isometric embeddings of Riemannian manifolds have long been studied in differential geometry. Of particular interest is a well known theorem (Janet-Cartan) [14,15] which

states that if the embedding space is flat, then the minimum number of extra dimensions needed to analytically embed a Riemannian manifold is d , with $0 \leq d \leq n(n-1)/2$. The novelty brought by Campbell-Magaard theorem is that the number of extra dimensions falls drastically to $d = 1$ when the embedding manifold is allowed to be Ricci-flat (instead of Riemann-flat).

IV. NEW EMBEDDING THEORIES AND THE NEED FOR MORE GENERAL THEOREMS

The increasing attention given to the Randall-Sundrum models [16,17] in which the embedding manifold, i.e., the bulk, corresponds to an Einstein space, rather than to a Ricci-flat one, has raised the question whether Campbell-Magaard could be generalized and what sort of generalization could be done. It was conjectured that if the Ricci-flatness condition were replaced by the requirement of the embedding space being an Einstein space, then a result similar to Campbell-Magaard theorem would hold. That this conjecture is in fact a theorem was recently shown by Dahia and Romero [18]. Specific classes of embeddings, such as those of Einstein spaces into Einstein spaces were established [19]. This was the first extension of Campbell-Magaard theorem and more were to come.

A following question to address was whether further extensions of Campbell-Magaard theorem were possible. This led to an investigation of embeddings in spaces that are sourced by dynamical matter fields. One of the simplest forms of matter is that of a scalar field, so it was natural to consider this case first. It was then proved that any n-dimensional analytic Lorentzian or Riemannian space can be locally and isometrically embedded in a (n+1)-dimensional analytic manifold generated by any arbitrary self-interacting scalar field. As a corollary one can prove that any Lorentzian or Riemannian n-dimensional analytic manifold can be embedded in a (n+1)-dimensional space which is a solution of the vacuum Brans-Dicke field equations [20].

In seeking higher levels of generalization one is led to consider the more general situation

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of embedding spaces whose Ricci tensor is arbitrary. Pursuing this idea a bit further a third extension of the Campbell-Magaard was finally established. A theorem was proved which asserts that any n-dimensional semi-Riemannian analytic manifold can be locally embedded in a (n+1)-dimensional analytic manifold with a non-degenerate Ricci tensor which is equal, up to a local analytic diffeomorphism, to the Ricci tensor of an arbitrary specified space. As an application of this theorem, embeddings of Ricci-flat spacetimes into five-dimensional Friedmann-Robertson-Walker models were obtained in ref. [21].

To conclude we would like to point out that insofar as five-dimensional embedding theories are metric it appears to be of relevance to allow the embedding spaces to have different geometrical properties, which must ultimately be determined by the dynamics of the theory in question.

Therefore, generalizations of the known embedding theorems might reveal crucial in building new higher dimensional models. On the other hand, from the standpoint of epistemology, it is rather illustrative to see how modern theoretical physics can be a source of new ideas in mathematics and geometry.

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